THE ARITHMETIC TEACHER

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Which Way Arithmetic?

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Something is happening to the way arithmetic is being taught in the elementary school. This fact is so obvious it need not be documented. A few minutes spent comparing the books of today with books published just a few years ago will show this to be true. What is producing this change?

The Old Arithmetic

Arithmetic is emerging from a period in which it was broken up into small "learning bits." It was fragmented to make the subject easier to learn. After all, so the reasoning went, complex skills are developed by drilling on one little part of the skill before undertaking the task of mastering another part of the skill. This position was taken because the processes (addition, subtraction, multiplication, and division) were emphasized almost to the exclusion of thought processes and because, formerly, it was thought that, "Teaching the fundamental facts of arithmetic is a habit-forming process. What we desire to do is to form bonds in the child's nervous system between stimuli and responses."[1]

Such a concept of how children learn arithmetic resulted in emphasizing proper drill procedures and in analyzing the subject of arithmetic into unit skills. The process of breaking arithmetic into "little pieces" is illustrated by indicating some of the unit skills frequently mentioned in methods books which discuss the teaching of addition.

1) Easy column addition (sums under 10)
2) Column addition (sums in the teens)
3) Zeros in column addition
4) Higher decade addition
5) Column addition (sums in the twenties)
6) Column addition (sums in the thirties)
7) Addition practice preliminary to carrying in multiplication
8) Addends of two or more digits
   a) no carrying
   b) carrying in tens place only
   c) three digit answer
   d) carrying in tens and one place
9) Addends of unequal length
10) Addends of three or more digits

Splitting the processes into small items was the prevailing mode less than twenty years ago and, in many instances, it is still the mode today. A book used widely in the late 1930's and 1940's states, "More and more, primary teachers are realizing the importance of making a detailed analysis of the materials of the curriculum and the necessity of arranging a graded list of exercises through the use of which the pupils may gain a thorough knowledge

[1] Robert Lee Morton, *Teaching Arithmetic in the Primary Grades*, (New York: Silver Burdett Company, 1927), p. 35. (The author modified his position considerably in subsequent editions of his book).

of the subject-matter and acquire the desired habits."[2] Other instances of the fragmentation of arithmetic can be readily found in the literature. Notable among these, is the 80 skills in the division of common fractions found by F. B. Knight and his pupils.[3]

The Revolt Against the Old Arithmetic

Analyzing the processes of arithmetic into minute skills did not go on without vigorous protests from a number of educators. In 1937 Wheat wrote, "We have moved from the piecemeal type of instruction advocated by Pestalozzi in our teaching of reading and writing. We remain, however, in the Pestalozzian period of meticulous dissection in our teaching of arithmetic. In this, the twentieth century, our arithmetic is in the main in the early nineteenth century period of development. This subject has been analyzed for the pupil into a multitude of combinations, processes, formulas, rules, types of problems, etc., and the pupil is taught each in turn as a separate item of experience. Often, when he has completed the course, he knows only those parts that he can still remember, and they all seem to him as separate and unrelated combinations, processes, formulas, rules, and types of problems to be solved. Finally, when his memory for these separate items fails him, he has nothing left to carry into his adult world but the remembrances of a series of meaningless, uninteresting, and unpleasant experiences that his classes in arithmetic seem to have provided him."[4]

In 1941 T. R. McConnell wrote, "In the case of arithmetic, attempts to psychologize the subject appear to have damaged it both logically and psychologically. By decomposing it into a multitude of relatively unrelated connections or facts, psychologists have mutilated it mathematically, and, at the same time, by emphasizing or encouraging discreteness and specificity rather than relatedness and generalization, they have distorted it psychologically. They have obscured the systematic character of the subject, and have created a doubtful conception of how children learn it. Furthermore, the practice of connectionism in arithmetic leads almost inevitably to immediate emphasis on rapid and accurate computation rather than on the development of the ability to think quantitatively."[5]

But in spite of vigorous protests, the forces of the specific habit advocates were too strong. Arithmetic was divided and divided into so-called "learning units" until it was hard for the learner and the teacher to see that the subject really contained fundamental ideas—ideas which enabled people to meet daily life situations efficiently. Instead, arithmetic became a machine, a tool subject, with facts to be memorized by the carload and specific rules to be kept in mind to solve problems that had been typed for the pupil. With such a program, it is little wonder that many people rapidly lost faith in arithmetic as a subject containing much that is of educational worth. A certain minimum amount of arithmetic was a necessity for an educational program emphasizing living in a technical society; after all, the children would be required to buy groceries at some future date. Beyond this minimum, however, many educators saw little or no value in arithmetic. Arithmetic had come to a "pretty pass" because of a faulty theory of learning and a misguided philosophy of education.

[2] Robert Lee Morton, *Teaching Arithmetic in the Elementary School*, (New York: Silver Burdett Company, 1937), Volume 1, p. 204.

[3] G. M. Ruch, F. B. Knight, E. A. Olander, and G. E. Russell. *Schemata for the Analysis of Drill in Fractions.* University of Iowa Studies in Education, Volume X, Number 2 (Educational Psychology Series, No. 3), 1936.

[4] Harry Grove Wheat, "The Psychology and Teaching of Arithmetic," (Boston: D. C. Heath and Company, 1937), p. 156.

[5] T. R. McConnell, "Recent Trends in Learning Theory: Their Application to the Psychology of Arithmetic," *Arithmetic in General Education* Sixteenth Yearbook, The National Council of Teachers of Mathematics, (New York: Bureau of Publications, Teachers College, Columbia University, 1941), p. 275.

Some Ideas Producing a Change

Warnings, such as those quoted above, that the practice of breaking arithmetic into unit skills creates "a doubtful conception of how children learn" are gradually taking effect. As a result teachers are now more willing to put into effect certain elements of a new psychology of learning and to place less emphasis on certain principles and practices of a specific habit psychology. Space does not permit a detailed discussion of this point even though it is important enough to merit it. Only a few of the principles and ideas which are not a part of the specific habit psychology and which are being more readily accepted today—in theory at least, if not in practice—will be identified.

Insight. Today teachers intuitively accept the fact that insight is important in the teaching of arithmetic. To this end, they feel that the ability to visualize and imagine is the key to developing thought processes. Consequently methods of today place an emphasis on such things as ten's blocks to develop the concept of place value. Once the thought processes have been stimulated by the manipulation of the blocks, teachers encourage the child to substitute symbols as a more efficient way to think about quantities. Through this means insight is gained into the structure of the decimal system of numbers. Today, thought and action are recognized as being "in the same world" and both are necessary to attain a real understanding of the fundamentals of arithmetic.

Growth Processes. There is greater realization on the part of today's teachers that learning is a growth process and that habituation is the final stage in learning. This means that in the initial stages of learning a process immature and inefficient responses are acceptable. The problem in teaching becomes that of changing the child's response to one that is more acceptable by adult standards. Furthermore, it is being more widely accepted that drill does not contribute a great deal to this growth process.

Thinking. It is recognized, today, that understanding, insights, and meanings help children to think about quantity and to produce rapid and accurate computers. The days of "this is the way you get the answer" teaching are *passé*. Today the teacher encourages the child to structure ideas and symbolize them so as to produce the arithmetic power that can only develop through meanings and insight.

Experience. Teachers believe, today, that abstractions and generalizations develop out of many experiences with objects and that symbolization of these experiences is a later stage in the learning of arithmetic. As a result, more emphasis is being placed on visual aids in the teaching of arithmetic. The modern visual aids program is not the old concrete to the abstract program of yesterday. Now teachers realize that the symbols $5+4=9$ are not necessarily abstract for the child. If the child realizes that they symbolize a situation in which a group of 4 joins a group of 5 to make a group of 9, $5+4=9$ is not abstract, although the child has made an abstraction.

While other changes in the fundamental concept of how children learn arithmetic could be given, these will serve to illustrate the basic reason for the changes in arithmetic over the past two decades. By and large, these changes have been accepted as desirable by teachers but they have not always known how to make maximum use of some of the new principles to produce a better arithmetic program. Some teachers have reluctantly accepted the new ideas. Still others have rejected them entirely because they cannot, or will not, break patterns of thought and practice that they have developed over a term of many years, even though evidence may not support their position. Such a stand, of course, puts the convenience of the teacher above the welfare of the child.

Using New Principles of Learning

While such principles as those referred to above are being more readily accepted today, there is still real reason to believe that a more ready acceptance of these principles would benefit the child a great deal. To this end, some rethinking must be done in the field of arithmetic. The processes of arithmetic are still atomized to a great extent. A study of the teaching division in the fourth, fifth, and sixth grades in most any textbook will supply sufficient evidence to this effect. To break the process of division into some thirty unit skills, not counting the basic division facts, is an all too common practice. This is a hangover from the days of specific habits. Furthermore, teachers still place too much faith in drill as a stimulant to the learning process. This too has been retained from the days of a specific-habit psychology.

While arithmetic has changed, it has not gone far enough. Newer principles of learning have not been consistently applied. Arithmetic is still caught in the coils of tradition in spite of some rather sound evidence that changes are overdue. Only a few of these overdue changes will be considered in this article.

Problem solving. Volumes have been written about problem solving. Problems have been analyzed and children have been tested, interviewed, and analyzed, but the difficulty still remains. Some children still want to add when they encounter a "how many more are needed" problem. Why?

In the initial stages, children learn by showing them things, and by manipulating objects. This showing, doing, and acting eventually are put aside as inefficient ways of responding to situations and a symbolic response is substituted for what the child has observed happening in the situation. This is the "thought follows from action" principle which has been creeping into the teaching of arithmetic. Let us illustrate how this operates in learning how to solve problems.

Children are taught that when groups are joined the process of addition is used to find the size of the resulting group. This is illustrated in actual teaching by taking a group of blocks, objects, markers, etc. and showing just what happens. The sequence of pictures below (Figure 1) illustrates what a teacher might do to show a child what $6+3=9$ means. Note that *actions* are used. The child sees one group joining another group. After a few more experiences with such situations the child associates addition with the joining of two groups. Furthermore, he should associate $6+3=9$ with all those situations in which a group of 3 joins a group of 6. Because of

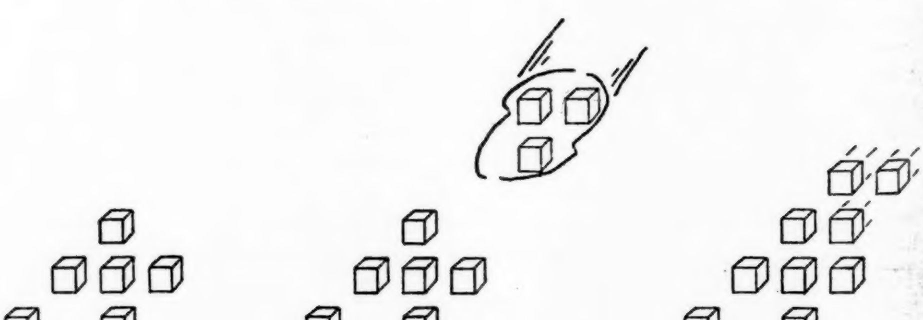


FIG. 1

these experiences with the joining actions, the child recognizes an addition situation when he sees it. The words used (how many in all; sum; etc.) are not used to identify the situation. It is the total event —mainly the actions; that which is happening—that identifies the situation for the child. Thus, regardless of whether the situation involves birds flying, pennies earned, cookies given, etc., all "joining situations" are additive situations and are symbolized by an expression such as $6+3=9$.[6]

Now assume that the teacher has firmly implanted in the mind of the child that the joining of two groups (regardless of the words used) is a characteristic of addition. (For our purpose we will call it an additive situation.) The child is then ready to solve a different kind of additive problem. Consider this one: Mary had 6 cents. Her mother gave her some more money and then she had 9 cents. How much money did Mary then have? The sequence of pictures below (Figure 2) shows what happened.

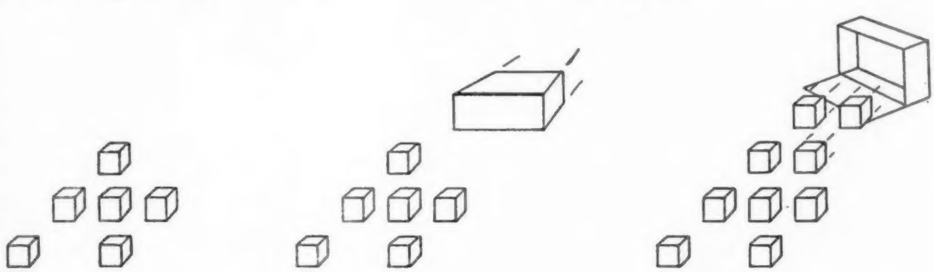
Notice that two groups joined. Previously —through actions and words—the teacher has told the child that such "joining" situations are "addition problems." However, standard practice dictates that we tell the pupil to subtract to get the answer. In view of the child's first experience with joining groups, is it any wonder that children want to add to solve the "how many were added" and the "how many are needed" problems?

In this instance the teacher has been unconsciously clever. She has joined two groups BUT SHE DIDN'T TELL THE CHILD THE SIZE OF THE JOINING GROUP. She then told the child that it is a subtraction problem.[7] Confusion? Most certainly! Trouble with problem solving? Most certainly!

The fact is that this is an additive situation and should be so presented to the child if we are to avoid confusion. Only by so doing can we avoid having the child say, "The words are like adding but you subtract." In these situations the child should be taught to symbolize, (1) what happens in the situation by writing an equation $6+N=9$, (2) to do the necessary subtracting by writing the numbers in computational form; i.e. $\begin{array}{r}9\\ \underline{6}\end{array}$, and (3) to interpret this result in terms of the original situation by writing $6+3=9$. Through this method the concept forming situation (6 apples and 3 apples are how many apples) serves as a basis for further learning. The procedure outlined here would prevent the confusion that is evident

[6] The situations just discussed are used to introduce addition. They will be called the concept-forming situations.

[7] For purposes of this discussion, we assume that the child has been taught that separating a subgroup from a group is the chief characteristic of a subtractive situation.



Mary has 6 blocks

Her mother gave her some more. (Blocks in box joining the 6 blocks.)

She now has 9 blocks. How many blocks did mother give Mary?

FIG. 2

when the child works the above problem by writing

$$\begin{array}{r}6\\+3\\\hline 9.\end{array}$$

Note that the answer was written as an addend. This is a very natural thing to do in the light of the child's past experiences. Such reactions are indications of the child's thought processes in these problems. He thinks of them as additive situations.

There is yet a third kind of additive situation—an instance in which two groups join. Jon had some blocks. His mother gave him 3 blocks. Jon then found that he had 9 blocks. How many blocks did Jon have originally? The situation is shown in Figure 3. Here again the teacher can "out-fox" the child and herself unless she is careful. Note that, again in this instance, two groups are joining (additive) but that the group that is being joined is not known while the joining group (3 blocks) is known. If the child has learned his initial lesson well, he should call this addition.

In order to avoid confusion the child should be taught, (1) to write an equation showing what happened in the situation by writing $N+3=9$, (2) to do the necessary computing by writing

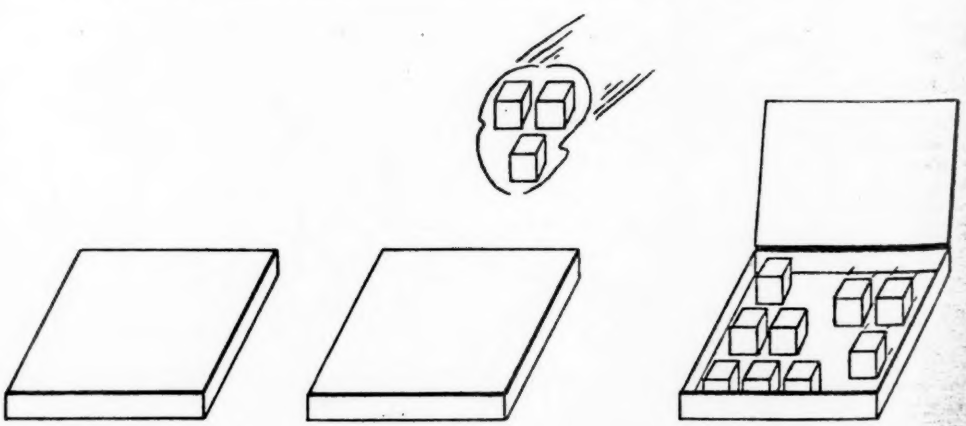
$$\begin{array}{r}9\\3\\\hline 6\end{array}$$

and (3) to interpret this answer in terms of the original problem by writing $6+3=9$.[8]

From this discussion it is evident that there are three kinds of additive situations at least two of which are commonly taught in the second and third grade. These problems should be taught so as to emphasize their similarities. This is in agreement with the newer psychology. They should not be taught by rule (To find how many more are needed you subtract.). Rule teaching and "cue" teaching is a principle of the older psychology which is finding less favor in the modern classroom.

To teach arithmetic the new way requires a new frame of mind on the part of the teacher; it requires a new set of techniques. Arithmetic can be taught this way; it should be taught this way.

[8] The last two additive situations correspond to Harold Moser's "twilight zones" in division. See Harold Moser, "Can We Teach to Distinguish the Measurement and Partition Idea in Division?" *The Mathematics Teacher*, Vol. XLV (February, 1952), pp. 94–97.



Jon had some blocks. (Pictured as hidden in the box.)

His mother gave him 3 more blocks. (3 blocks joining an unknown number of blocks.)

Jon then found that he had 9 blocks. How many blocks did Jon have at first?

FIG. 3

Processing Numbers. Traditionally arithmetic has been thought of as number processing only. That it is a means of symbolizing experiences is rather a foreign thought to most programs of arithmetic. This, of course, is a result of the connectionistic psychology that has dominated the subject for so many years. Here, as in problem solving, teachers must rethink their instructional program.

On the previous pages we have already illustrated how, in the past, arithmetic was analyzed into unit skills. In order to illustrate how arithmetic instruction should retreat from this practice, a small segment of the process of addition of integers will be discussed.

A recommended procedure for teaching children how to add two two-digit numbers consists of the following.[9]

a) Teach the addition of exact tens—$30+20=$
b) Teach the addition of an exact ten and another two digit number—$30+41=$
c) Teach the addition of two two-digit numbers without carrying—$32+46=$
d) Teach the addition of two two-digit numbers with carrying—$36+28=$

Several questions arise concerning such commonly practiced procedures. First, is it necessary to break the process into four separate steps? Is McConnell's statement that "Evidence is accumulating slowly but surely which reveals that when the learner understands the number system and the operations which its structure permits, he has developed insight into arithmetical processes which makes instruction and drill on each variant or every specific 'fact' unnecessary,"[10] applicable here? Thirdly, when the child adds $\begin{array}{r}21\\ \underline{45}\end{array}$ is he really thinking about adding 21 and 45 or is he thinking about adding 1 and 5; then 2 and 4; placing the answers in close proximity? To state this last question another way; is the child *really* getting the basic idea of adding two-digit numbers while learning to add without carrying, or is he acquiring some machinery which must be modified at a future date?

There is reason to question the usual approach to adding two two-digit numbers. Let us assume that the child has a good background in the number system, developed through a well-planned program of instruction. Why not let the child in on the whole secret at once by teaching him to add such numbers as 46 and 17 the first time a two-digit addition problem is encountered? This procedure requires him to think about adding 46 and 17 not just two instances of basic number combinations. The examples, $21+45$, $30+67$, and $30+20$ all become interesting special cases of an important idea—adding two two-digit numbers without carrying involved. The child now has the general idea. He does not need drill on each variant of the general case—the whole idea—but he does need to see how they all are special cases of a larger whole. Maybe, arithmetic is better taught and easier learned if the teacher emphasizes systems of ideas and variants of the system—differentiation—rather than trying to build a system by teaching parts first and letting the child in on the secret at the final stage of the learning process.

Division, Divison furnishes another excellent but lengthy illustration of how a process has been divided into parts, thereby losing the general idea. There is no reason why the concept of subtracting successive multiples of the divisor should not be the basic idea around which the whole division program is built. At a very early stage in the process a child could then find how many 12's in 50 by any one of the following methods:

[9] Herbert F. Spitzer, *The Teaching of Arithmetic.* (New York: Houghton Mifflin Company, 1954), pp. 107–111.
[10] *Op. cit.*, p. 276.

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12)50 |            12)50 |
  12  |  1           24  |  2
  --                 --
  38  |              26  |
  12  |  1           24  |  2
  --                 --  | ----
  26  |               2  | 4 R2
  12  |  1
  --                 12)50 |
  14  |                48  |  4
  12  |  1             --  | ----
  --  | ----            2  | 4 R2
   2  | 4 R2
```

The central idea used here is to subtract as many multiples of the divisors as you can. If you don't take away enough, then take away some more multiples of the divisor. This is all there is to division (measurement) for the remainder of the instructional program. Nothing more; nothing less. As the child becomes more proficient with the machinery, he will shorten the computation, but the thought processes are not varied throughout the whole instructional program.

Same Answers—Same Thought Processes. It is prevalent in arithmetic to think that because two operations give the same answer, the thought processes must be the same. This misconception is illustrated by the usual instructions accompanying the following problem. Mary has 24 apples which she wants to place in 4 boxes with the same number of apples in each box. How many apples should she place in each box? Common practice instructs the child to take $\frac{1}{4}$ of 24 since this is the same as $24 \div 4$. This is an error from two points of view. In the first place, if the child is taught that the partitive-division problem given above can be solved by taking $\frac{1}{4}$ of 24, he will associate division with the process of taking $\frac{1}{4}$ of 24. Later on he must learn that $\frac{1}{4}$ of 24 is multiplication, and he must symbolize it by writing $\frac{1}{4} \times 24$. Confusion? Of course!

In the second place, the mental set for a partitive-division situation is not the same as it is in a $\frac{1}{4}$ of 24 situation. In the partitive-division situation the center of attention is on the equality of ALL the groups. This is as it should be. Any anxiety about the situation ends here. There are 6 in each group and all the groups have the same number. The problem is solved. In the $\frac{1}{4}$ of 24 situation the attention is centered on the equality of all groups as in partitive division, BUT now there is an additional step. The attention is next centered on one of the groups as satisfying the conditions of the problem. Thus, if Johnny gets $\frac{1}{4}$ of the 24 apples, he pays attention to the equal groups (of 6) first and then centers his attention on just one of the 4 equal groups. The mental set in a situation symbolized by $\frac{1}{4}$ of $24 = N$ is different from the mental set in a situation symbolized by $24 \div 4 = N$[11]

The Professors' Errors

Over the years a number of terms have developed in the terminology of the teaching of arithmetic that could well be dropped. True, the retention or the elimination of these terms would have little or no effect on the arithmetic curriculum, but their very existence indicates the degree of confusion that exists in (1) the terminology of arithmetic, and (2) the analysis of the thought processes which the symbols represent. A few of the more common of the errors will be considered.

The Abstract Multiplier. Johnny put six apples in each of three boxes. How many apples did Johnny put in the boxes all together? In a common analysis of this situation, it is contended that three is an "abstract number"; the six is "concrete"—6 apples—and the answer (18) is "concrete"—18 apples.

The diagram in Figure 4 pictures the situation. Here we see the six apples in each of the three boxes. Which numbers are the so-called "concrete" numbers? In the first place, there are three groups of apples. These groups are just as "concrete" as the apples themselves. In the

[11] The partitive division problem under consideration should be symbolized by writing $24 + N = 4$. Measurement problems should be symbolized by writing $24 \div 4 = N$.

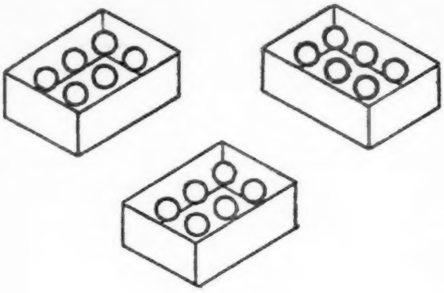

FIG. 4

second place, there are *6 apples per group*; not just six apples as claimed by some analysis of such problem situations. A closer analysis of multiplicative (and divisive) situations shows that they always contain a rate number. The 6 apples per group is a rate number.

Simple multiplicative and divisive situations always contain three numbers. One number indicates the number of groups under consideration. A second number indicates the number of objects per group—a rate number. The third number indicates the total number of objects that are under consideration—the number of objects in all of the groups. All three must be considered as "concrete" numbers or as "abstract" numbers. There is no logical alternative.

This analysis shows that there is no reason to call the multiplier abstract, in fact, it is in error to do so. Mathematically, all the numbers should be abstracted from the situation for computational purposes and we should write $3\times 6=18$. Having obtained the 18 it should then be interpreted in terms of the problem situation. To write 3×6 apples $=18$ apples is mathematically incorrect because denominations should not appear in a mathematical equation. Physicists frequently do put the denominations in the equation. If this practice is followed, then the dimensional analysis shows that the equation should be written as

$$3 \textit{ groups} \times 6 \textit{ apples per group} = 18 \textit{ apples}$$

The Three Meanings of a Fraction. Here we have the confusion caused by having the same word for the symbol and the idea. This confusion is avoided in the case of integers by calling the symbol "2" a numeral and the idea it stands for a number.

The symbol "$\frac{2}{3}$" is commonly called a fraction. The idea it stands for is a combination of partitive division and a counting (cardinal) number. Thus, to take $\frac{2}{3}$ of 24 we divide 24 into three equal parts (partitive-divisive) and then count out 2 of the 3 groups. This is the fraction idea and the symbol for the combination of these two ideas is $\frac{2}{3}$.

Now, unfortunately, we use the same symbol, $\frac{2}{3}$, to stand for other ideas. Thus it may represent the ratio of two to three. But the ratio of two to three is not a fraction. It is not partitive division plus a cardinal number. So, it is not another meaning of a fraction, but it is another idea for which the same symbol is used as for the fraction idea.

The symbol "$\frac{2}{3}$" is also used to indicate a division. In this case the same symbol stands for yet another idea, but it is not the idea of a fraction.

This analysis shows that the term "three meanings of a fraction" is a loose verbage. *There are three ideas represented by the same symbol.* This is common practice in the English language as it is in mathematics. The word "wind" has at least two meanings. However, note that we do not speak of the two meanings of "wīnd" or "wĭnd." Many other examples could be given of multiple meanings of a symbol.

Concrete Number and Abstract Number. Number is an idea and as such it cannot be concrete. So in reality the term "concrete number" is a contradiction. It would be much better to say that two apples is a representation of the number two. The number is not concrete but its representation may be concrete.

There term "abstract number" is a good example of a term not needed at all. Number is an abstraction. Hence, the term

"abstract number" says, in effect, an abstract abstraction. This is excess verbage. The adjective "abstract" may as well be dropped. The symbol "3" then is spoken of as a numeral and the idea it represents is a number. Abstract number? Why not use "number" only?

In Conclusion

Like the american automobile, new models of the arithmetic program are coming out. The new arithmetic models have been coming out for the past few years. The underlying reasons for the new models in the arithmetic program is that teachers are slowly but surely changing their concept of how children learn arithmetic. This requires new ideas, new techniques, and new equipment. Furthermore, it requires some deep thought on the part of all of those who are responsible for any part of the program for educating the elementary school child.

Some teachers will say, "These new ideas are too hard. We can't teach them to kids." This may be true, but it should be kept in mind that for adults to relearn is often more difficult than it is for the child to learn. To relearn requires the checking of reactions that may have been habituated for years and years, and what is more difficult to break than a well established habit? Children do not have these response patterns for arithmetic habituated. They don't have the same learning problem the teacher has. Hence, it does not necessarily follow that what is hard for the teacher is also hard for the child.

The features of the 1960 model for arithmetic are being made by the teacher in the classroom during the years 1955–1960. The horsepower of the engine to be placed in this model is being determined in these same classrooms during the same period. Whether this horsepower and the features of the new model will be good or bad, adequate or not adequate, will depend, to a great extent, on how much we know about the thing we are working with—the child and the learning process.

EDITOR'S NOTE. Dr. Van Engen has thought a good deal about several basic considerations in Arithmetic. He has made an excellent beginning in rethinking the implications of newer principles of learning and their application to a modern program of teaching and learning arithmetic. Other leaders ought to be pioneering and still others, particularly those who are teaching the teachers of arithmetic, should be experimenting with new modes of organizing content and learning. Do you agree with Mr. Van Engen's analyses and conclusions? Is it more important in the new arithmetic that children learn to distinguish "measurement" division from "partitive" division? Who will carry forth the argument?

The National Council of Teachers of Mathematics

Sheraton-Park Hotel, Washington, D.C.

DECEMBER 27-29, 1955

SPECIAL ARITHMETIC MEETINGS

Wednesday, December 28, 1955—Continued

10:45 A.M.–12:15 P.M. ELEMENTARY SCHOOL SECTION *Burgundy Room*

Presiding: MARY L. GRAU, Public Schools, Montgomery County, Maryland

Arithmetic for Those Who Excel

FOSTER E. GROSSNICKLE, State Teachers College, Jersey City, New Jersey

The Intangibles in Arithmetic—How They Are Developed

JOHN R. CLARK, New Hope, Pennsylvania

"HERE'S HOW I DO IT" SECTIONS

This is a series of informal conferences for the discussion of mutual interests and problems. Audience participation is desired.

2:00 P.M.–3:00 P.M. ARITHMETIC CONFERENCE *Burgundy Room*

Presiding: NANETTE BLACKISTON, Public Schools, Baltimore, Maryland
Consultants:
EDWINA DEANS, Public Schools, Arlington, Virginia
ELLA MARTH, District of Columbia Teachers College, Washington, D. C.
IRENE M. REID, District of Columbia Teachers College, Washington, D. C.

3:15 P.M.–4:45 P.M. ARITHMETIC LABORATORY *Caribar Room*

Leaders:
ELDA L. MERTON, Chicago, Illinois
ANN PETERS, State Teachers College, Keene, New Hampshire

Thursday, December 29, 1955

9:15 A.M.–10:15 A.M. DEMONSTRATION LESSON IN ARITHMETIC *Continental Room*

Presiding: EDITH A. LYONS, Assistant Superintendent of Schools, Washington, D. C.
Teacher: EVELYN BULL, Janney Elementary School, Washington, D. C.

9:30 A.M.–12:00 noon SHOWING OF MATHEMATICS FILMS AND FILMSTRIPS *Mural Room*

SECTIONAL MEETINGS

10:30 A.M.–12 noon ELEMENTARY SCHOOL SECTION *Burgundy Room*

Presiding: ANN PETERS, State Teachers College, Keene, New Hampshire
Development of Reading Skills in Arithmetic
R. L. MORTON, Ohio University, Athens, Ohio
Some Emotional Factors in The Learning of Arithmetic
ROBERT S. FOUCH, Florida State University, Tallahassee, Florida

10:30 A.M.–12:00 noon TEACHER EDUCATION SECTION *Franklin Room*

Presiding: HOUSTON T. KARNES, Louisiana State University, Baton Rouge, Louisiana
Mathematics Background Needed by Teachers of Arithmetic
DANIEL SNADER, University of Illinois, Urbana, Illinois
In-service Teacher Education—Whose Responsibility?
KENNETH E. BROWN, U. S. Office of Education, Washington, D. C.

Thursday Afternoon

1:30 P.M.–2:30 P.M. GENERAL SESSION *Continental Room*

Presiding: CHARLOTTE W. JUNGE, Wayne University, Detroit, Michigan
The Continuity and Sequence of Mathematics Concepts—Grades One through Twelve
PHILLIP S. JONES, University of Michigan, Ann Arbor, Michigan

DISCUSSION GROUPS

These discussion groups will take direction from the ideas presented during the preceding General Session. They will indicate the concepts to be developed at each level and present ideas as to how these concepts may best be developed.

2:45 P.M.–4:15 P.M. ELEMENTARY SCHOOL GROUP *Burgundy Room*

Presiding: GLENADINE GIBB, Iowa State Teachers College, Cedar Falls, Iowa
Consultant: HAROLD MOSER, State Teachers College, Towson, Maryland

Ten Years of Meaningful Arithmetic in New York City

Laura K. Eads
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During the past 10 years meaningful arithmetic has become the generally accepted method of teaching and learning in the elementary schools of New York City. This means that some 20,000 supervisors and teachers are now making varying efforts to help children *think* mathematically and to *understand* the mathematics they learn. It means also that these supervisors and teachers are learning or have learned the meaning of arithmetic themselves.

Sporadic efforts to develop a program of meaningful arithmetic in New York City have been made since the beginning of the century. As recently as 1930 Saul Badanes, then a principal in one of the city's elementary schools, prepared materials for teachers and children in the primary grades. In the Teacher's Book he stated: "We show the teacher how to lead the pupil, by means of his own self-activity, to a clear understanding of and insight into the learning process. We help the teacher fix permanently in the mind of the pupil the results of his own experiences and investigations, not by mere monotonous repetitions and verbal associations, but by means of associations resulting from the thoughtful organization of each unit of instruction." "All devices, all objective illustrations should be placed in the hands of every child." "We make drill work conscious and deliberate, namely, after the pupil has gained insight into the concept of number and the meaning of operations with number through a long series of observation and self-activity." (Macmillan Company.)

The Badanes program was inaugurated in a few experimental schools but, despite the enthusiasm of supervisors and teachers in these schools, meaningful arithmetic was not adopted in other schools and did not become officially accepted. By 1945 there were only a few supervisors and teachers who practiced any of the Badanes procedures of teaching arithmetic.

Planning a Curriculum Project in Arithmetic

The present program of meaningful arithmetic was planned in detail as part of a total curriculum program. The results of research in child development, in learning, and in learning arithmetic were studied and analyzed. Prevailing over-all curriculum programs and programs for teaching arithmetic were studied with a view toward outlining ways to reconcile theory and practice.

Ten years ago a detailed "Outline of Curriculum Project in Arithmetic" was prepared to get reactions from curriculum committees, from individual teachers and supervisors, and from the administration. Some excerpts from this outline follow:

> Twenty years ago when the *Course of Study and Syllabus in Arithmetic for Elementary Schools* was projected and developed, the computational aspects of arithmetic were emphasized. This was an outgrowth of the widely expanding testing movement which, in arithmetic emphasized the computational aspects rather than the development of concepts, understandings, and thinking about numbers and processes.
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> Teachers emphasized the same thing that the tests emphasized—getting a correct answer. Since right answers could be arrived at through memorization, teachers emphasized the memorization of rules for getting the right answer most quickly rather than mathematical thinking; thus came about an era of rote learning in arithmetic with its emphasis on drill and more drill on number facts and on examples.
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> As might have been expected the children didn't learn why they performed the tricks

they did and thus their ability to apply these number operations to problems or to real situations declined markedly during these years, and yet, improvement in computation did not result.

Next, came diagnostic and remedial programs in arithmetic. The child's errors in computation were discovered and he was helped to re-memorize whatever he forgot or to memorize what he hadn't learned about facts and processes. More and more time was allotted to arithmetic teaching and re-teaching—and still the results were disappointing.

Present-day knowledge of how children learn and how they develop concepts indicates that *meaning* must be developed before the computational aspects are stressed. The importance of the social aspects of arithmetic—the social situation to introduce a mathematical concept as well as social applications in problems—is also recognized.

The child not only is ready for arithmetic when he first comes to school, but, actually, has been ready for some time before this. But the kind of arithmetic he is ready for is very different from the kind considered beginning arithmetic by many schools and teachers.

It must not be presumed that because a child counts he understands the numbers involved. It cannot be presumed that he knows the mathematical value of the numbers. It certainly cannot be presumed that he has a working knowledge of any of the numbers involved.

Meaningful arithmetic necessitates some reorganization of the arithmetic topics to be learned. Heretofore, the sequence and the grade placement of topics was determined by what was known regarding the facility with which children could learn number facts and processes through memorization and by rule. Where *thinking* and *understanding* in arithmetic are to be developed, number relationships and number concepts will be developed some time before number facts are learned or memorized.

It is important at this time to plan an experimental program designed to explore and study issues and problems in developing meaningful arithmetic. Such a plan should aim to coordinate the various efforts being made throughout the city. It should eventuate in the development of arithmetic materials which will be helpful to teachers and to supervisors in organizing an arithmetic curriculum program based upon an understanding of how children learn, as well as upon an appreciation of arithmetic as a systematic science.

An Exploratory Study

In September 1945 an exploratory study of children's mathematical thinking and of teaching procedures was organized in one school. We began with two classes, Grade One and Grade Two. Teachers of these classes began to teach mathematical relationships, (a) by pointing up mathematical situations whenever they occurred during the school day, (b) through units of work, and (c) through planned systematic instruction. Children were questioned in class groups and individually. The thinking of gifted children was particularly noted.

During the exploratory study many needs arose. Among these were the need for: teacher and supervisor reorientation, curriculum materials, teacher training materials, concrete materials for children, re-interpretation of the role of classroom experiences, re-allocation of curriculum content, re-allocation of teaching time, re-evaluation of procedures for group teaching, re-evaluation of achievement tests in arithmetic, re-evaluation of the concept of grade placement of content in arithmetic in terms of the varying needs of children.

Preparation of Experimental Curriculum Bulletins

The preparation of curriculum materials was necessarily our first consideration. These were needed to get official approval for experimentation, as well as for use by supervisors and teachers of experimental classes. The 1937 New York State bulletin, *Mathematics for Elementary Schools*, served as the basis for preparing New York City's experimental bulletins. The first of these was: *Arithmetic: Kindergarten—Grade Three* (1947, out of print). Other bulletins prepared as the experimental program proceeded to higher grades were: *Mathematics: 4* (1951), *Supplement to Mathematics: 4* (1952), *Mathematics: 5–6* (1952).

Evaluation of Experimental Curriculum Bulletins

The next step in the curriculum research was an evaluation of the point of view, the content, the sequences, and the procedures suggested in the experimental bulletins, beginning with the bulletin for the primary grades. A program of evaluation

was planned on two levels: first, experimental research in selected schools; then, experimental tryout on a city-wide basis.[1]

During the year 1947–48 experimental research involved classes in Grades 1–4. Experimentation was gradually extended to include higher grades, as the city-wide experimental program was being implemented in lower grades (one grade at a time).

As the research proceeded evaluations and curriculum adaptations were made cooperatively by teachers and supervisors. Sometimes these were "on-the-spot" adaptations by children and teachers in the classroom. Parents too were involved. Engineer fathers devised teaching aids. Mothers suggested experience situations and sent concrete materials—beads, buttons, felt, etc. They helped their children make purchases and measurements and keep records of these. These were planned, evaluated, and used in the classroom. The records provided data for developing addition, subtraction, multiplication, division, and measurement.

Learning How Children Think

Children were studied in class groups, in smaller groups, and individually. Teachers and supervisors discussed and tried a variety of procedures for finding out how well children at varying levels of maturity understood and could use the mathematics they were taught. As experimentation proceeded supervisors and teachers agreed upon the following conditions as necessary for developing thinking and for finding out what and how children think.

1. *Provide time and opportunity for children to tell or to show what the mathematics means to them.* This requires that teachers avoid giving clues, asking "leading questions," or channeling children's thinking in standard patterns. Questions or comments such as the following are suggested: What do you mean? What did he say? Can you tell it in another way? Tell me more. Tell it in *your* words. Show it with your beads.

   This was difficult for some teachers. At first they hurried the children—listening to children or waiting for them seemed a waste of time. When children could not show or explain their thinking some teachers commented: He knows but can't explain it. I know she knows; she knew it yesterday. She doesn't understand what you want; let *me* ask her. It wasn't easy for teachers to learn how long its takes for understanding to be developed.

   Often, when insight came to a teacher, it came suddenly. The day a teacher appreciated how *understanding* differed from *rote verbalization* was a rewarding day for both that teacher and her supervisor. Usually these teachers began then and there to help other teachers "understand too."

2. *Accept children's own expressions.* Emphasis on complete sentences, correct grammatical expression, precise adult mathematical terminology, etc., is likely to interfere with children's thinking out of mathematical solutions.

   Teachers found it difficult at first to hear how children really expressed themselves. Their first records of children's words were edited records. Teachers hesitated to quote incomplete sentences or ungrammatical expressions. After teachers learned to hear what children really said, they began to hear what their colleagues too said. They reported that few adults used complete sentences when thinking mathematically, and that incomplete sentences and ungrammatical expressions were common.

3. *Expect more mature thinking* (that is, more precise and more advanced levels of thinking) *from more mature children*

[1] Assistant Superintendent Arthur Hughson described the city-wide program in the November, 1955 issue of THE ARITHMETIC TEACHER.

than from less mature children. This requires that teachers know what to expect from children at various levels of ability.

Differences among children in type and quality of mathematical thinking were found to be very marked. Children were asked to tell what some common mathematical terms meant. Typical responses of children in one fifth grade class to "What does *pair* mean?" follow: Least mature children—A pair of something. To add. Pair of shoes. Something we eat. Hair. Parents. Two. Bright children—Twins are a pair. Two of a kind. The double of one thing. Two things that go together or people that belong together. Two matching things.

Supervisors and teachers were amazed at the great disparity in understanding and ability to generalize among the most mature and least mature children in a class.

4. *Talk less and listen more.* This requires that teachers stop often to find out what children at different ability levels are thinking and learning. It requires that teachers plan frequently to listen to slow children, "average" children, and bright children—that they listen carefully in order to sense the meaning behind each child's long, involved explanation.

Teachers were concerned at first about the "waste of time" and found it difficult to adjust to a talk-less, listen-more program. Classroom management also needed to be considered. Children were accustomed to raising hands whenever they wished; in some classes children called out answers to "help" a slow-thinking child. Teachers soon learned that children hesitated to continue thinking out solutions when other children waved hands. Teaching children to "think along" with the child making an explanation, to listen without raising hands, took some time. An impatient child was encouraged to listen carefully and attentively by being asked what he had to add to the explanation, or if he could say it in a different way, etc.

5. *Differentiate between what you teach and what various children learn.* This requires continuous appraisal of the learning of children at varying ability levels.

Mathematical thinking takes longer then most teachers realized. After a teacher "taught a lesson" children were questioned. Teachers reacted in various ways: One teacher insisted again and again: But I taught that: He *should* know it. Several teachers felt uncertain and asked: But what *do* I teach? Every now and then a teacher tried to put the responsibility on the children: They come from broken homes. Their parents teach them arithmetic at home. But gradually teachers learned about their children, how long it takes for concepts to develop, how children differ from one another, how much "luckier" their children were than they (the teachers), etc.

From Grade 5 on, teachers discovered to their dismay at first, but with pride later on, that brighter children soon were able to think better mathematically than their teachers, that challenging these bright children to higher levels of thinking required that the teacher learn more and more mathematics. As one teacher said: "It's hard work, but I love it."

Developing, Evaluating, and Using Concrete Materials

During these 10 years of experimentation, children, teachers, and supervisors have devised, improvised, tried out, and rejected a variety of types of concrete materials to help make mathematics meaningful. There is very little that hasn't been thought of or used by some teacher in this city.

The following excerpts from a Progress Report for 1947–48 exemplifies this:

> This phase of arithmetic teaching was new to the teachers and to the children as well. But it intrigued them. Everyone became active. Beads were strung, cards were cut, sticks were bundled together by tens, wooden frames were put together, pie plates were divided into fractional parts, demonstration tables were devised and improvised, milk bottle tops were collected, checkers were "found," milk containers were cut and colored, boxes or books or statuettes became "props," bags for objective materials were made, shelves for storing material were devised, charts were prepared.
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> Materials appeared from everywhere in all guises and disguises. They were pulled out of long unused closets. They were brought from home by the children. They were purchased by the teachers. They were prepared by teachers and by children. Shop teachers and children in the upper grades were called in to help. Rows of tens were devised from erstwhile costume jewelry, babies' toys, kindergarten beads, shell macaroni, lima beans, small jars, clothes pins, etc. The "beads, balls, bottles" arithmetic was in full swing.
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> Once begun, the emphasis on objective materials was difficult to put to constructive use. The teachers were gradually helped to realize that the manipulation of materials, in and of themselves, was not arithmetic for the children. Thinking by the children was re-emphasized.

Gradually, through the years, teachers and supervisors have come to feel that concrete materials are essential in all grades. They feel, however, that concrete materials are needed only until children at various ability levels, and the teacher also, *understand* the mathematics. Less mature children need to use concrete materials long after the most mature children are able to think out mathematical relationships referring only to symbols.

Widespread Participation in Developing and Evaluating Curriculum Materials

During the first years, experimental research involved selected classes in selected schools. Experimental tryout, one year at a time beginning with Grade One, was city-wide.

In preparing and evaluating curriculum materials, however, every effort is made to involve as many supervisors and classroom teachers as feasible in a city as large as New York. Some procedures used in preparing a bulletin for teachers in Grade 3 (to accompany and interpret the 1951 Course of Study) follow.

1. Tentative material had been prepared some years ago and used in mimeographed form with study groups of supervisors who adapted the materials for use in their schools. Problems faced by teachers as they used such materials were discussed in the study groups. Mathematics Coordinators used preliminary materials in their districts throughout the city.
2. A committee of teachers and supervisors prepared a tentative outline of content (8 topics) for the third year. As topics were elaborated they were tried out and critically evaluated by cooperating supervisors and teachers. Reports of procedures and experiences found successful in the classroom were requested. Every suggestion and criticism was considered significant. In discussing these reactions, consideration was given to why the reviewer may have reacted as she did, as well as to what she actually said.
3. Revised materials were re-submitted to cooperating teachers and supervisors to find out if the revision was adequate. This procedure is continued until we are satisfied that we are preparing the kind of material supervisors and teachers want and can use. Thus, when a bulletin is finally issued it has actually been in use for some time.

Children in experimental classes were tested at least once a year. Existing standardized tests were used. This was important since teachers and supervisors wanted to be sure that children were growing in ability to compute and to solve paper-and-pencil problems, while meaning and the thinking out of relationships and solutions were stressed. From the beginning, nearly all of the experimental classes exceeded the national norms in both computation and problem solving.

Supervisors asked for additional instruments to use in evaluating concepts in Arithmetic. Dr. J. Wayne Wrightstone, Bureau of Educational Research, directed the preparation of the *New York Inventory of Mathematical Concepts* for use in each of the elementary grades.

To help teachers evaluate pupil growth and progress "in those aspects of mathematics in Grade 3 that cannot be measured by the usual paper-and-pencil test: measurement and fractional parts," Associate Superintendent Florence S. Beaumont, Division of Elementary Schools, directed the preparation of a booklet entitled *Observation Inventory of Mathematical Understanding: Grade Three.*

The program of curriculum development in the area of arithmetic may be briefly characterized as action research involving: first, supervisors and teachers in selected experimental schools; then, supervisors and teachers on a district-wide basis; then, city-wide experimentation with tentative curriculum materials; then, try-out and evaluation by cooperating teachers and supervisors of material preliminary to publication; and finally, publication of bulletins to accompany and interpret the 1951 *Course of Study in Mathematics: Grades K-6. Mathematics: 1–2* (1955) is the first of a new series of bulletins to be published.

EDITOR'S NOTE. The New York City program in arithmetic has received a good deal of attention because it is a serious attempt to reorient a very large group of teachers. It is not easy for teachers to change their thinking and schoolroom procedures after these have been well established. The New York story contains good advice for all of us. We must accept our pupils as children and encourage them to grow instead of imposing an adult pattern upon them in their early learning stages. Dr. Eads has been one of the prime developers of the New York City program of meaningful arithmetic. We shall watch how this develops during the next ten years. What significant new programs are being created in other cities and smaller communities about the country? Let us print the story.

Labeling Answers to Arithmetic Problems

ANNA ULLRICH
Washington School, West Allis, Wisconsin

A MUCH DISPUTED QUESTION which frequently confronts the elementary school teacher in marking answers to problems in arithmetic is: "If the answer has no label but is otherwise correct, should it be marked *right* or *wrong?*" In the hope that certain guiding principles in the matter of labeling answers might be formulated, the writer made a questionnaire investigation which involved approximately 250 school people ranging from teachers to superintendents in Wisconsin and some 30 authors of books, tests, and articles in periodicals outside the state. The Wisconsin group also included instructors in normal schools, colleges, and universities.

This investigation is the outgrowth of a previous study of the literature on the teaching of arithmetic and an examination of scores of textbooks, standardized tests, and the answer keys for both. In the literature there was not a single indication of guiding principles regarding labeling of answers to problems. Many inconsistencies were found. The questionnaire for the present investigation was based on the inconsistencies previously found. The questionnaire had two main sections: I—GENERAL and II—MISCELLANEOUS and two auxiliary sections; III—PLEASE SUGGEST A RULE WHICH ELEMENTARY SCHOOL CHILDREN WOULD UNDERSTAND AND IV—REMARKS.

In the following copy of the section, "I—General," illustrations of the items are omitted for conservation of space.

I—GENERAL. After each question, please *underline* "YES" or "NO".

A. In general, do you think that the answer should be labeled to be considered correct?. . . . . *YES*. . . *NO*

B. Would you mark an answer correct if the child omitted any kind of label whatever including such signs as $, ¢, ° (for degrees) and %, except in a problem specifically calling for a label as in the case of listing recipe ingredients?. . . . . *YES*. . . *NO*

C. If the problem asks, "*How many?*" of a certain thing or unit of measure expressed in the problem, would you mark an answer *correct* if there were no label?. . . . . *YES*. . . *NO*

D. If the problem asks, "*How much?*" "*What amount?*" or the like, with reference to a certain thing or unit of measure (not including money) expressed in the problem, would you mark an answer *correct* if there were no label?. . . . . *YES*. . . *NO*

E. Do you think that a pupil should be required to label an answer if the question asked is, "*How many?*" of a certain *unit of measure*, such as, bushels, gallons, pounds, miles, etc., but that the label may be omitted if the question asked is, "*How many?*" of a certain *thing*, such as, trees, cows, marbles, pupils, books, etc.?. . . . . *YES*. . . *NO*

F. In a problem involving the use of the dollar sign (or the cents sign in case the amount is less than a dollar) if the question asked is, "*What amount?*" "*At what price?*" "*What is the cost?*" "*How much?*" or the like, would you mark the answer *correct* if the dollar sign or cents sign were omitted?. . . . *YES*. . . *NO*

G. Would you mark *correct* an answer not labeled if the question asked is, "How far, high, deep, long, wide, tall, etc.?" if the unit assumed in the answer is that which is expressed in the problem?. . . . . . . *YES*. . . *NO*

H. In a problem dealing with area, if dimensions are given in the same unit of measure, would you mark an answer *correct*, if it were not labeled?. . . . . *YES*. . . *NO*

I. Would you mark *correct* an answer not labeled if the question asked is, "What is the length, width, height, depth, thickness, distance, etc.?" if the unit assumed in the answer is that which is expressed in the problem?. . . . . *YES*. . . *NO*

J. While it has been indicated that this set of questions concerns only "problems," this one question refers to "examples." If among a set of examples made up of abstract numbers for testing computation there is an example having the dollar sign, would you mark the answer *correct* if the dollar sign were omitted?. . . . . *YES*. . . *NO*

II—MISCELLANEOUS. Please *underline* whatever answer or answers you would consider correct for each of the following problems:

1. A certain village has a population of 1,500 while the neighboring village has a popula-

tion of 750. *What is the combined population* of the two villages?

2,250 or 2,250 people

2. *What is the average speed* of an airplane that travels 1,000 miles in 5 hours?

200 or 200 miles or 200 miles per hour

3. *What is the average speed per hour* of an airplane that travels 1,000 miles in 5 hours?

200 or 200 miles or 200 miles per hour

4. Mrs. Jones used 8 eggs in a cake. *What part of a dozen* is this?

$\frac{2}{3}$ or $\frac{2}{3}$ dozen

5. The distance through the center of the earth is about 8,000 miles while the distance around the earth is about 25,000 miles. *How much greater is the distance* around the earth?

17,000 or 17,000 miles

6. Leo weighs 72 pounds. His baby brother weighs 18 pounds. *How many times* as much as his baby brother does Leo weigh?

4 or 4 times

7. Nancy has a ribbon which is $\frac{5}{6}$ of a yard long. *What is the length of the ribbon in inches?*

30 or 30 inches

8. The front blackboard has an area of 30 square feet while the side blackboard has an area of 48 square feet. *How much larger* is the side blackboard?

18 or 18 square feet

9. Our school purchased 121 new books for 11 rooms. *What was the average number* of books bought for each room?

11 or 11 books

10. Normal body temperature is 98.6°. Billy's temperature was 2.4° above normal. *What was Billy's temperature?*

101 or 101°

The following two charts showing results of the investigation correspond to the two sections of the questionnaire.

RESULTS OF ARITHMETIC QUESTIONNAIRE

I—GENERAL

Items	Number of teachers who checked items			Per cent of teachers	
	Yes	No	Total	Yes	No
A	219	58	277	79	21
B	80	199	279	29	71
C	167	111	278	60	40
D	82	200	282	29	71
E*	152	111	284*	54	39
F	22	241	263	8	92
G	71	199	270	26	74
H	30	231	261	12	88
I	64	193	257	25	75
J	65	196	261	25	75

* Considered ambiguous by 21 teachers (7 per cent).

II—MISCELLANEOUS

Answers checked as acceptable	Checks	Total	%
1. 2,250 2,250 people	172 198	370	46 54
2. 200 200 mi. 200 mi. per hr.	36 84 252	372	10 22 68
3. 200 200 mi. 200 mi. per hr.	47 146 200	393	12 37 51
4. 2/3 2/3 doz.	156 205	361	43 57
5. 17,000 17,000 mi.	52 255	307	17 83
6. 4 4 times	121 216	337	36 64
7. 30 30 in.	91 243	334	27 73
8. 18 18 sq. ft.	36 257	293	12 88
9. 11 11 books	121 213	364	41 59
10. 101 101°	54 266	320	17 83

In the "General" part it is to be observed that not every teacher checked every item. Hence the variation in totals. This "throws off" slightly the accuracy of the "Yes" per cent column for purposes of comparison. Allowing for this slight inaccuracy, the per cents of "Yes" responses in the "General" section follow this rank order:

1.	A	79%	6.	G	26%
2.	C	60%	7.	I	25%
3.	E	54%	8.	J	25%
4.	B	29%	9.	H	12%
5.	D	29%	10.	F	8%

(Item "E" was considered ambiguous by 7 per cent.)

As the charts indicate, the greater number favor labeling generally. A good many (60%) would permit omission of a label for an answer to a problem asking for "How many?" of a certain unit of measure or of things. Over half (54%) would require a labeled answer if the question asks for "*How many?*" of a certain *unit of measure* but would permit omission of a label if the question asks for "*How many?*" of a certain *thing* as trees, cows, etc. Twenty-nine per cent would permit omission of all labels including \$, ¢, °, and %.

In the II "Miscellaneous" section preference is given to labels. In Number 1, dealing with population, there is little difference. Numbers 2 and 3 show a decided preference for labels, in both cases "miles per hour." No great difference is found in Number 4 ($\frac{2}{3}$ or $\frac{2}{3}$ dozen), the unlabeled answer having 43% and the labeled answer 57%. Number 5 which asks "How much greater is the distance, etc." has 17% for no label and 83% for a label. Number 8 shows considerable difference, 12% accepting no label for area while 88% prefer the label. Equally interesting is the response for Number 10 where 17% would permit no label while 83% favor a label for "degrees." Other comparisons may be made by reference to the chart.

For Part III "Suggest a Rule" and Part IV, "Remarks" there were responses that overlapped considerably. Of all responses only 250 could be more or less "categorized."

Considering Parts III and IV jointly, the following outline summary gives some kind of idea of the trend of responses. (Figures represent the number of answers rather than the per cent.)

I. "Labels should be required"—Total 89
   A. "Label all answers"—61
   B. "Label for clear thinking and understanding"—15
   C. "Answers should be given in complete sentences"—7
   D. "Require children to label all answers to avoid confusion"—7
II. "All problem answers should be labeled unless the question explicitly states the label"—40
III. "The answer to the problem should answer the question asked in the problem"—39
IV. "Deduct only a certain amount of credit if the label is missing"—25
V. "Labels are necessary only for units of measure (including money)"—14
VI. "Consider correct every answer without a label except answers dealing with \$, ¢, %, and degrees." (Some stated only \$ and ¢)—12
VII. "Consider problems a reading experience. Interpret answers"—8
VIII. "Avoid rules"—6
IX. "Habits in arithmetic are more important than rules"—4
X. "Do not mark an unlabeled answer wrong. Call attention to the need of a label"—4
XI. "Consider the individual needs of the children"—4
XII. "It is not fair to mark an answer wrong if the work is right but the label is missing"—2
XIII. "Labeling depends upon the object of the lesson"—2

(Several called attention to redundancy

concerning "C" of "General." For instance, this comment was made about illustration (1) under "C": "The answer to the question is 27. To say 27 pupils, in this case, is redundant." The problem was: "Miss Smith has a class of 15 boys and 12 girls. How many pupils has she?")

Summary of Findings

In this summary it is assumed that the "answer" referred to is always numerically correct.

Judging from results of checking the "General" questionnaire section one may safely assume that:

1. There are differences of opinion regarding the extent to which arithmetic problem answers should be labeled but that the majority (79 per cent) of those who answered the questionnaire favor labeling generally.
2. Sixty per cent would mark "correct" an unlabeled answer if the problem asks "How many?" of a certain thing or unit of measure expressed in the problem.
3. Over half (54 per cent) think that a pupil should be required to label an answer if the question asked is "How many?" of a certain *unit of measure*, such as bushels, gallons, pounds, miles, etc., but that the label may be omitted if the question asked is "How many?" of a certain *thing*, such as, trees, cows, marbles, pupils, books, etc.
4. More than one fourth (29 per cent) would mark an answer "correct" if the pupil omitted any kind of label whatever including such signs as $, ¢, ° (for degrees) and %, except in a problem specifically calling for a label as in the case of listing recipe ingredients.
5. More than one fourth (29 per cent) would mark "correct" an unlabeled answer if the problem asks, "How much?", "What amount?" or the like with reference to a certain thing or unit of measure (not including money) expressed in the problem.
6. More than one fourth (26 per cent) would mark "correct" an answer not labeled if the question asked is, "How far, high, deep, long, wide, tall, etc?" if the unit assumed in the answer is that which is expressed in the problem.
7. One fourth (25 per cent) would mark "correct" an answer not labeled if the question asked is, "What is the length, width, height, depth, thickness, distance, etc?" if the unit assumed in the answer is that which is expressed in the problem.
8. One fourth (25 per cent) would mark "correct" an answer omitting the dollar sign if among a *set* of examples made up of abstract numbers for testing computation there is an example having the dollar sign.
9. About one eighth (12 per cent) would mark "correct" an unlabeled answer to a problem dealing with area if dimensions are given in the same unit of measure. (Illustration: 10 *feet* by 13 *feet*)
10. Only 8 per cent would mark "correct" an answer omitting the dollar or cents sign in a problem in which the question asked is, "What amount?", "At what price?", "What is the cost?", "How much?", or the like.

The "Miscellaneous" questionnaire section indicates a fair degree of consistency with the "General" section. With reference to the type of answer considered acceptable one may safely assume that:

1. Generally there is a preference for a labeled answer regardless of how the question is stated.
2. If a problem asks, "What is the population?", the preference is a label, such as "people" (or rather, "persons").
3. If an answer is a fraction (for example in the answer to "What *part* of a dozen, etc.") the preference is to have the label of the unit, as, "$\frac{2}{3}$ dozen" rather than merely "$\frac{2}{3}$".

4. The preference for expressing "speed" is "miles per hour" rather than an abstract number of merely "miles."
5. A labeled answer to a problem asking such a question as, "How much greater is the distance around the world, etc?" is far more preferred to an unlabeled one.
6. If the question asked is, "How many *times* as much, etc?" a labeled answer is preferred to an abstract number.
7. If the question asked expresses the unit in such a question as, "What is the length of the ribbon in inches?" a majority of teachers prefer a labeled answer.
8. Even though the unit of measure is explicitly expressed in such a problem as, "The front blackboard has an area of 30 square feet while the side blackboard has an area of 48 square feet. *How much larger* is the side blackboard?", the majority of teachers favor a labeled answer.
9. In a problem asking for the "average number" of a certain thing, a labeled answer is preferred.
10. The majority of teachers prefer a label for degrees if the question asked is such as, "What was Billy's temperature?"

PARTS III "Please suggest a rule, etc." and IV "REMARKS" indicate that:

1. Very many teachers believe in a blanket rule: "Label all answers."
2. Many teachers believe that the label *may be omitted* in the answer if the "label" is expressed explicitly in the question.
3. Many teachers believe that the answer to the problem should answer the question asked in the problem (indicating acceptability depending upon the discretion of the teacher).
4. A considerable number of teachers advocate deducting a certain amount of credit if an answer is not labeled; they would not mark the answer entirely wrong.
5. A considerable number of teachers feel that problem-solving is not a mere answer-getting process and that stress should be placed not so much upon how the answer is stated as upon how the child has reasoned in his work.
6. Some teachers feel that labels are necessary only for units of measure, including money.
7. Some teachers would consider correct every answer without a label except answers referring to $, ¢, %, etc.
8. Some believe in avoiding rules but prefer to train children to state answers in such a way that the question in the problem is answered.
9. A few advocate complete sentences for answers so that there is no doubt in the child's mind as to what is the "label."
10. Individual differences and abilities must be considered in marking answers in arithmetic.

Conclusions

The investigation concerning the labeling of arithmetic problem answers has not *settlee* anything as for a definite guide in the matter.

There is much evidence that this matter presents a difficulty not only to elementary school teachers but also to all who are deeply interested in teaching problem-solving.

Textbook answer keys omitting labels for certain problem answers may be far better understood as a result of this investigation (perhaps the outstanding result) inasmuch as a majority of teachers have indicated that they would consider "correct" an unlabeled answer if the question asks, "*How many*?" of a certain unit explicitly expressed in the question. Close examination of a textbook, the answer key for which sometimes omits labels, will reveal that it is the "How-*many*" type of problem (in which the "label" as a rule is explicitly expressed in the question part) which is the one for which the answer key gives *no* label.

Much of what the authors of arithmetic textbooks know about acceptable answers, the elementary teachers *do not* know. The average, teacher, for example, does not know why one author omits labels in most cases while another author gives labels for all of the answers that can be labeled. (How many elementary teachers, for example, consider the matter of "redundancy"?)

Taken for granted that problem-solving is not mere answer-getting, the answer, nevertheless should answer the problem. But the child (who knows even less about "redundancy," for example, than does his teacher) may be very much puzzled if he should attempt to analyze why the answer key omits labels for some problems and not for others.

Even though the teacher may refuse to be disturbed over such a matter as "answers," regardless of how deeply she is concerned about teaching the child and not the subject, teaching problem-solving as a reasoning process, etc., etc., she still does not escape the inevitable testing, and she may have no control over rules for checking answers.

In the last analysis, it is the child who is affected fundamentally. It is he who is learning to *reason* in problem-solving in arithmetic, but it is also he who is being marked either justly or unjustly. In the present state of confusion concerning the matter of labeling answers to problems, who is right?

The cooperative effort of teachers, administrators, supervisors, and textbook and test authors apparently is needed to work out a solution.

Since diversity of opinion is bound to exist even within a school system, a common understanding might be agreed upon for guiding principles to be listed in an arithmetic handbook (1) to aid the teacher and the pupil in understanding the textbook answer key and (2) to serve as *authority* for answer keys for problems prepared for testing within the system.

EDITOR'S NOTE. Miss Ullrich has found that there is no real agreement among teachers and writers in the matter of "labeling answers" to problems in arithmetic. It is disturbing that we have no generally accepted principles for guidance of teachers, editors, etc. Miss Ullrich's investigation is a good starting point for some responsible group that will seek to establish guiding principles for use by writers, editors, and teachers. The editor suggests that a committee of writers and teachers, sponsored by The National Council of Teachers of Mathematics, undertake the task and that the resulting guide be made available to school people generally.

Arithmetic We Need, Grades 3, 4, 5, 6, and 7. *Teaching Arithmetic We Need*, Grades 3 and 4, Buswell, Guy T., William A. Brownell, and Irene Sauble. Ginn and Company, 1955.

The first five books of this series of Arithmetics and the first two of a promised set of MANUALS are attractive volumes and bear witness to the careful work for which their authors are so well-known. We have come to expect that textbooks in arithmetic will have gay bindings, an attractive format, lavish use of color, and problems and illustrations that appeal to children. The series *Arithmetic We Need* fulfills these expectations. In addition, there are features that greatly increase the usefulness of these texts to a busy teacher. For example, a page in the sixth grade book is as follows: the heading is

AT ZENO'S

At the bottom of the page is a picture of Zeno's Fruit Stand which is the scene of the problems. Underneath the heading, underscored in red, is the direction "*Tell which estimate after each problem seems best and why.*" The key words "Estimating answers" are in small italics at the upper right and (O) following the key words indicates that the work that follows should be oral. A number of problems follow with three plausible answers after each and from which the pupil is to select the best one. At the bottom of the page are the words

"Now go back and *write your work for each problem.*" An experienced teacher will not require such explicit directions but the beginning teacher will find that they lead him to pay the attention that the authors desire in order that oral discussion be an essential part of class work in arithmetic.

The MANUALS amplify the work further. Each gives an introduction telling how to use the Manual, then an overview of the program for the grade in question, the arithmetic background that the textbook assumes, and then the details of the textbook for that grade, chapter by chapter. Each textbook page is reproduced on a smaller scale than in the text with a commentary telling the pupil's objectives, the pupil's background, the teacher's preparation for the lesson even to the properties that are needed, the pre-book lesson, the book lesson containing suggestions for dealing with the slow learner and with the rapid learner, and finally "Differentiations and Extensions" which suggest further activities. Following these suggestions, a classroom would soon become a place in which you are constantly reminded that you need arithmetic and that you use it everyday.

The textbooks made ample provision for practice in computation.

The series has outstanding values.

On the negative side, the following items seem unfortunate to the reviewer. In the *Manual* for Grade 3, p. 72, a picture labeled NUMBERS CUPBOARD shows a bulletin board with a horizontal row of eight hooks and hanging from each a string of ten beads. Under the strings of beads are numbers reading from left to right, 10, 20, 30, 40, 50, 60, 70, 80. The title reads "Organizing and labeling groups of objects by tens helps children learn how 2-place numbers are written." This suggests that in counting a bowlful of beads, the children string them in groups of 10 and then count the groups. But in the picture, a child points to the label 20 and nothing indicates that this number represents the 10 beads over the 10 *and* the 10 beads above the 20. In Grade 4, the study of Roman numerals begun in Grade 3 is reviewed by looking at the numerals on a clock face. It is a minor point, but the clock face in the picture has IV for 4. What is the probability that a pupil will find 4 in that form on the next Roman numeral clock

face he sees? Isn't it more likely that the clock will be marked IIII? It would seem wise to give the pupil both forms with the statement that IIII is seldom seen anywhere except on a clock face. In Grades 5 and 6, the authors have stated "We usually say that any number that contains a decimal fraction is a *decimal*." They then restrict the word "fraction" to a common fraction and condense "decimal fraction" to *decimal*. This practice is widespread, but none the less unfortunate. It tends to foster the idea that "*fractions*" and "*decimals*" are radically different. On page 105 of the text for Grade 5, there is a serious blunder. The question is how many inches are there in $\frac{3}{4}$ of a yard. The explanation reads:

1 yd. = ? in.; $\frac{1}{4}$ of $36 = ?$; $\frac{3}{4} = 3 \times 9$ or ?

The same pattern is repeated later on the same page. Let us hope that the final statement is revised to be $\frac{3}{4}$ of $36 = 3 \times 9$ or ?

VERA SANFORD

The following pamphlets are "Educational Service Publications" available from The Extension Service, Iowa State Teachers College, Cedar Falls, Iowa. They are reviewed by Miss Angela Pace.

ISSUE NO. 2. *Arithmetic: For Developing an Understanding of Place Value*, H. VanEngen, 1946. 10 pages, 10 cents.

This bulletin presents procedures for developing an understanding of place value. It shows how such simple materials as tongue depressors, sticks, squared paper, money, an abacus, and a wall chart can be used effectively to help children understand the positional character of our number system.

ISSUE NO. 3. *Arithmetic: The Teaching of Fractions in the Upper Elementary Grades*, H. VanEngen, 1946. 27 pages, 10 cents.

This bulletin describes procedures for teaching fractions meaningfully in the upper elementary grades. Basic fraction concepts to be developed in these grades are presented. The author then describes many types of experiences which the pupils should have if they are to grow in their ability to use fractions effectively in quantitative situations. Many practical suggestions for teaching of the fundamental operations with fractions are offered. The use of diagrams in the initial stages is stressed and illustrated in detail. Teachers will find this part especially valuable.

ISSUE NO. 5. *Arithmetic: Using a Ten in Subtraction*, D. Banks Wilburn, 1947. 10 pages, 10 cents.

This bulletin is of particular value to the teachers of the third and fourth grades. It describes in detail procedures which might be used in teaching children in these grades how to use a ten in subtraction. Specific suggestions are given for guiding children's experiences in using a ten meaningfully. Concrete objects are recommended for use in the initial stage. Less concrete procedures are described for guiding the children gradually from this initial stage to more mature levels of thinking.

Hindu-Arabic Numerals

VERA SANFORD
State University Teachers College, Oneonta, N. Y.

THE NUMERALS WE USE seem so simple and practical that we take them for granted. We tend to belittle them because they are familiar. On the other hand, we are painfully aware of the difficulties children have in learning to use them. So it is appropriate to survey the development of Hindu-Arabic numerals, not because we will at once relay the information to our pupils, but because we do our best teaching when the subject the children are investigating is one that commands our respect and admiration.

It is the purpose of this article to discuss the important features of our numerals and the history of these numerals.

Significant Features of Hindu-Arabic Numerals

Base 10

When a culture has developed to a point where large numbers must be named and recorded, we find a consistent use of the number ten. Thus Egyptian numerals have symbols for one, ten, one hundred (10×10), one thousand (10×10×10) . . . and numbers are made up by repeating these symbols i, e., two tens, five ones for twenty-five. This dependence on ten may be explained by the use of the fingers to show numbers as is indicated by the word digit (Latin *digitus*, a finger) meaning the numerals 1–9.

The use of ten as a basic element of a system of numerals is subject to variation. The Roman intermediate symbols for five ones, five tens, etc., provide one instance of this. The Babylonians, in astronomical work, used a grouping by sixties for very large numbers and by sixtieths for fractions. This is a system that still survives in our division of an hour into minutes and seconds. Thus a number we might write as 1,52 would mean 1×60+52 or 112. Traces of grouping by twenties exist in certain number names:

The English "three score and ten"
the Latin *undeviginti* (one less than twenty for nineteen)
the French *Quatre vingts* (four twenties for eighty). *Quatre vingt un* to *quatre vingt dix neuf* (for the numbers eighty one to ninety nine).

The idea of grouping by powers of ten (10, 10×10, 10×10×10) is basic to the Hindu-Arabic numerals. Thus the number 352 means 3 hundreds, 5 tens, 2 units. The pattern may be continued without limit.

Place Value and the Zero

When the number three hundred two is expressed in Roman numerals, it is written CCCII. In Hindu-Arabic numerals it is 302, the zero indicating that there are no tens. Zero is a necessary element in our system of numerals. It fills in the otherwise vacant places. There was no need for a zero in Roman numerals.

This is the really amazing but quite simple invention that we are using when we have children count pennies by grouping them in piles of ten and writing the number as the number of tens and the number of ones, or when we identify the hundred's place or speak of the ten's column.

History of Hindu-Arabic Numerals

There are three milestones in the development of Hindu-Arabic numerals. (1) It was about 300 B.C. that the ancestors of our numerals were used in the vicinity of the present Bombay in India. These numerals had symbols for the units, for

multiples of 10, for multiples of 100, etc. Thus to write the numbers from 1 through 999, you would need 27 different symbols. (2) It was about 600 A.D. that people were using a zero in these numerals making place value possible and making special symbols for numbers greater than 9 unnecessary. (3) The third milestone is the year 1585 when the first systematic account of decimal fractions was printed.

Prior to the time when zero appeared, Hindu numerals had the important characteristic that numbers from one to ten, multiples of ten, multiples of one hundred, etc. were not built up by repeating the symbols for one, ten, and one hundred, but were given distinct symbols of their own. This scheme had characterized other systems of numerals, the Hebrew ones for example.

The idea of a symbol for zero probably did not originate in India. In the case of Babylonian numerals of about 2000 B.C., the number 1,52 might mean $1\times60+52$ or $1+52/60$ or $1/60+1/60\times60$ according to the context. By 300 B.C., however, Babylonian records involving mathematics or astronomy make use of a symbol for zero. Thus the numerals for 11 followed by a zero would mean $11\times60+0$. Greek astronomers began using a symbol for zero also, probably borrowing the idea from the Babylonians, and when Hindu astronomers became acquainted with the work of these men, they incorporated the idea in their numerals. At this point, Hindu-Arabic numerals gained their great superiority over other systems for they coupled the idea of place value with a system that had distinct, separate symbols for the numbers 1–9 and for zero.

It was about 775 A.D. that important Hindu manuscripts dealing with astronomy and with mathematics were brought to Bagdad which was then the center of Moslem culure. The caliphs of Bagdad were active in collecting Greek and Hindu manuscripts and in having them translated into Arabic, and the Moslem universities became important centers of learning. It was at these universities that "wandering scholars" from Christian Europe learned the new numerals. Meantime, European traders dealing with Moslem countries had in all probability learned them also.

In Europe, Hindu-Arabic numerals made slow headway against Roman numerals. Arithmetic with the pen was in competition with arithmetic with counters. In the twelfth century and later, essays called *Algorisms* described these numerals and told how to use them in computation. The Italian, Fibonacci (1202), who had spent his boyhood in a Moslem city in North Africa, used these numerals in his *Liber Abaci* which was the most important book on mathematics produced in Europe in that century. Later, printed arithmetics sometimes gave both methods of computation. At times the new numerals were frowned upon. In 1594 merchants in the city of Antwerp were warned not to use the new numerals in contracts or in writing drafts on their bankers. This was nearly a decade after Simon Stevin of the nearby city of Bruges had published a short pamphlet on decimal fractions!

The extension of the idea of place value to include fractions put the finishing touch on Hindu-Arabic numerals. Stevin described these new numbers as sub-units calling them primes (tenths), seconds (hundredths), thirds (thousanths), etc. He marked these numbers by using a tiny zero in a circle to show units, a 1 in a circle for tenths, and so on. These symbols might be written above, below, or following the digit to which they applied or the one symbol for the place on the right might be written after the number. Stevin showed how the numbers were used in computation, establishing the validity of each process by converting the numbers and the result to common fractions. He showed, for example that if you were multiplying "seconds" by "thirds" the product would be fifths (hundredths times thousandths make hundred-thousandths). Stevin then tells how these numbers may

be used in surveying and in other computations involving measurements and he even hoped that they might be used in coinage. This hope was realized in the United States two hundred years later when this country was a pioneer in the matter of decimally divided coinage. In the introduction to the pamphlet, Stevin refers to the use of the base sixty in Babylonian fractions and says that he adapted it to his new numbers replacing 60 by 10. He says "If we may trust to experience, and we say this with all reverence to the past, the decimal and not the sexagesimal (60) is the most convenient of all progressions that exist potentially in nature."

Thus the Hindu-Arabic numerals had their origin in India and entered Europe from the Moslem culture. In all probability the zero was of Babylonian origin and came into India by way of the Greek writers. Finally, on the authority of the first person to write a detailed account of decimal fractions, this invention also stems from Babylonian sources.

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Cajori, Florian, *History of Mathematical Notations*, vol. I, Chicago, 1928. Encyclopedic, detailed, many facsimiles showing numerals of different countries and periods.

Smith, David Eugene, *Number Stories of Long Ago*, National Council of Teachers of Mathematics. Beautifully illustrated, may be used by pupils of the upper grades.

Smith, David Eugene and Jekuthiel Ginsburg, *Numbers and Numerals*, National Council of Teachers of Mathematics. Illustrated, an excellent source particularly for number names.

Smith, David Eugene, editor, *Source Book in Mathematics*, New York, 1929. Contains a translation of Stevin's work on decimals.

See also the histories of mathematics by Cajori, Sanford, Smith, and the *History of Arithmetic* by the late L. C. Karpinski. The last of these is unfortunately out of print.

For theories regarding the Babylonian connection with these numerals, consult *Science Awakening*, by B. L. van der Waerden, published by Noordoff, Groningen, Holland, 1954.

Topics and Questions for Further Study

1. Postage stamps and coins from countries that are predominantly Moslem show numerals—modern Arabic ones—that are closely related to ours, but that are not identical with them. List these numerals, and be on the lookout for pictures of street scenes in these countries to see the route numbers on buses, numbers on license tags and the like.
2. What numerals are used in China? in Thailand? in Japan?
3. If the base of a system of numbers is 2, the places in the numbers would have the values 1, 2, 2×2, $2\times2\times2$, $\cdots$. In such a system of numbers, what would be the value in our base 10 of a number 100_2? The little 2 at the right indicates the base. What would be the value of 1010_2?
4. How many symbols do you need to write numbers using the base 10? the base 2? the base n where n is a whole number greater than 1?
5. General Motors reported its net sales in 1954 as $9,823,526,291. Read this number. What base did you really use in pacing off this number? Complete this statement: In Hindu-Arabic numerals, the place value of each position in the number is a power of . . . .

EDITOR'S NOTE. As we increase our attention to meaning and understanding in arithmetic it becomes more important that teachers know about the nature and origins of the things we use. Our Hindu-Arabic system of numbers has often been called one of the most important inventions of the human race. Miss Sanford is one of our standard authors in the history of mathematics. Here she has detailed briefly some of the interesting facts and features about numerals and the way they have come to be used. References like *Numbers and Numerals* should be in every school library.

1956 Will Be Leap Year

"Leap Year" is mathematical. Normally, fixed holidays such as July 4, March 17, November 11, etc. come one day later in the week each year than in the preceding year. This is because $365 \div 7$ has a remainder of *one*. But in leap year $366 \div 7$ has a remainder of *two* and hence each fixed holiday after February will skip or *leap* an extra day in the week. Because of this "leaping" as associated with festivals and holidays in Britain the term "Leap Year" came into common usage several hundred years ago.

The Scarbacus or Scarsdale Abacus

LOUISE A. MAYER
Scarsdale, New York

THE "SCARBACUS," a project of a seventh grade group in Scarsdale, New York, grew out of a class discussion about types of counting devices. I said that I thought a counting device like an abacus should be one of the open end variety in order to give children a correct impression of number combinations. Young children in particular get a wrong impression from a counting frame that permits counting only to ten.

Bob Sanford, one of the pupils, said he could make an open end abacus and in a day or two brought in the framework here illustrated. It consists of a piece of wood into which are set ten dowel rods of uniform length and diameter. We decided to make one of the rods longer so that it would serve as a storage for the ring counters.

For ring counters we had to have something that would slip easily over the dowels. Hilary Karp suggested ivory crochet rings. Bob then painted the abacus black because he thought black and ivory looked well together. He also put a mark on each post to indicate the height of ten rings. On top of the longer storage post he fixed a knob which not only seals the storage but also can be placed on a post to represent zero or for a decimal point.

Meredith Baldwin named the instrument a "Scarbacus" because it was an abacus made in Scarsdale. Later copies have been made in maroon and white, the school colors. The red section of the posts extend as high as ten rings will cover.

Jeanette Cannon and Sheldon Ogilvy use the Scarbacus to demonstrate numbers.

The Scarbacus was originally designed for use in primary grades but I find it useful also in teaching place value in the number system, with directed numbers, and for "digit problems" in elementary algebra. It can also be used for demonstrating number systems other than our Hindu-Arabic system with the base ten. A blind boy in the third grade finds it both a pleasure and a real help in his number work. For primary grades an instrument with fewer posts would serve as well and would be easier to handle.

The materials for making a Scarbacus are readily available and the construction is not difficult. A boy in the junior high school grades who knows how to use simple tools can do it. The original Scarbacus was made in October, 1951. Its use has been demonstrated at teachers' meetings in Westchester County and at the Meeting of the Association of Mathematics Teachers of New York State. It was exhibited at the New England Institute held at Massachusetts Institute of Technology in August, 1954.

EDITOR'S NOTE. Pupils frequently learn a good deal in the development of a device such as the "Scarbacus." Later values depend much upon how a teacher leads pupils in their "discovery" of new ideas and principles. Several teachers have reported using an abacus in high school for illustration of adding and subtracting positive and negative numbers and numbers of bases such as two, eight, and twelve.

How Big Is a Billion?

Lauren G. Woodby
Central Michigan College, Mount Pleasant

Most of us will not have the experience of counting a million dollars, but, if we were to do it using new ten-dollar bank notes, the stack of bills would be about 40 feet high. How long would it take you to count out a billion dollars using one-dollar bills? Assuming that you are fairly skilled in the technique and can count at the rate of 4 every second, you could count $240 in a minute or $14,400 in an hour. It would take you nearly 2 forty-hour weeks to count out a million dollars. If you worked regularly for a year, you could count about 30 million dollars. You would be considering retirement before you finished counting the billion dollars!

A seventh grade girl, who was curious about the size of a million, counted a thousand kernels of dried corn and found that it amounted to approximately one cupful. She was surprised to find that a million kernels of this corn would amount to about 8 bushels. It was perhaps even more astonishing to learn that a billion kernels would be equivalent to about 8 thousand bushels. Certainly one outcome of this experience was a more concrete notion of the magnitude of a million.

The words "million" and "billion" are common ones, yet most of us have no concept of the magnitude of these numbers. One suggested procedure for making these numbers more meaningful to pupils in the elementary school is to have one child bring a pint jar of wheat and distribute portions to each pupil to count. Addition of the amounts counted would result in a better understanding of the numbers "hundred" and "thousand." Considerable experience in problem solving is possible. For example, one problem might be "How many grains of wheat are there in a bushel?". The resourceful teacher could capitalize on the opportunity to stress the meaning of the processes of addition and multiplication. Other problems will be suggested. The pupil who contributed the wheat might tell how many bushels were produced from the particular field, and the problem of finding how many grains of wheat were produced could be solved. A more complicated problem would be to find how many fields of this size it would take to produce a million bushels. Incidentally, the average yearly production of wheat in the United States is about one billion bushels.

Practice with large numbers obtained from counting objects is impressive because the physical size of the objects (or of the container) is closely associated with the resulting number. The experience of counting and handling the objects adds the element of realism that is often lacking in situations involving large numbers.

The size of crowds and the population of cities also can contribute to the meaning of thousand and million. For example, the University of Michigan Stadium seats nearly a hundred thousand people—twice the population of the city of Ann Arbor in which it is located. Any person who is a part of this crowd on a Saturday afternoon is likely to gain a better concept of the magnitude of this number. Ten such crowds would total a million people.

The population of the entire United States is about one-sixth of a billion.

It is interesting to note that the meaning of the term "billion" in Great Britain is quite different from the meaning of our "billion." In our system, a thousand million is called a billion and a thousand billion is called a trillion, but in the British system a million million is called a billion and a million billion is called a trillion.

How Many Children Are Here Today?

Mrs. Esther Instebo
Grade One, E. C. Hughes School, Seattle, Washington

Numbers seemed to become more real to my first graders last fall when we decided to use as counters the self-portraits we had made for open house. It was considered a privilege for one child to point to the portraits with a ruler while another tapped the actual person gently on the shoulder or head. This seemed to establish the one to one relationship so necessary in early rote counting. Each child watched intently to be sure he was tapped as his portrait was pointed out. The portraits were so real to the children that taking daily attendance became a true number experience very quickly. The inevitable first grade chart story of "We have——— boys, We have——— girls, We have ——— children." came easily because the children looked forward to counting their friends each day.

In addition to the usual practice of counting the boys, counting the girls, counting the total number, the absence of any class member soon pointed up the need for the use of subtraction. In finding out the total number absent the children soon discovered simple addition facts also. For instance, the absence of two boys and one girl not only made our room total less but also provided a simple number combination of 2 plus 1. As the days passed it was fun to work out number groupings with the children themselves and then to arrange their portraits to correspond. Thus we moved easily into counting by two's, three's, four's, and sometimes five's. We made a game of number stories, such as, "I see that 4 and 3 make 7." The corresponding subtraction fact "I see 7 boys, if 3 go away, 4 will be left" seemed to come naturally too. One day when 17 boys and 9 girls were present and the problem they made called for carrying it seemed the logical time to introduce them to the idea of beginning place value. We forgot boys and girls as such temporarily and counted children in groups of ten. From this regrouping evolved 2 tens and 6 more. Since then they have been encouraged to think in terms of ten and 7, ten and 1, and 2 tens (20) and 8.

As the children's understanding of number vocabulary began to develop meaning it provided repetition for their reading vocabulary as well. The keen interest in their number chart stories seemed to re-enforce the sight vocabulary in the reading program of such abstract words as "here," "there," "are," and "this."

As the year drew to a close it seemed apparent that the high interest level maintained in this number project throughout

the year was largely a result of the close relationship each child seemed to have with his own portrait and with those of his friends.

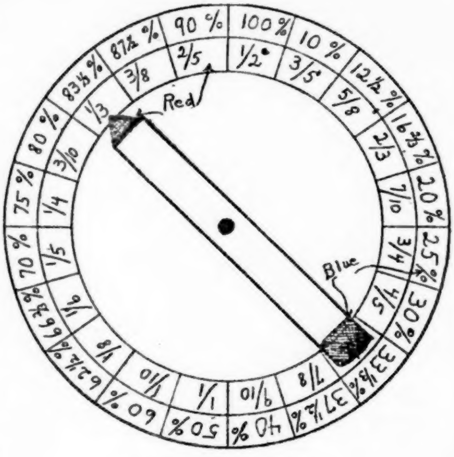
The portraits were made of orange-colored construction paper mounted on cardboard. The eyes and mouth were of cut paper. The hair was made of tulip yarn cut in appropriate styles and lengths and pasted in place.

EDITOR'S NOTE: Work with the number idea represented in things of great interest to children is usually very pleasurable to them. The "game element" inherent in the mode of teaching and guiding learning used by Mrs. Instebo is another factor of interest and motivation. Certainly it is much more fun to add real things in the early stages than to memorize the combinations of abstract number symbols and such meaningful learning uses a method of discovery which a pupil may project into new learning.

The Fracto-Percenter

WILLIAM B. ROYS
Scarsdale, New York

THIS DEVICE CAN BE MADE on heavy paper or "oak tag" and then pasted to a rigid background such as plywood or compoboard. The edges can be bound with masking tape. For classroom use it is suggested that the outer circle be twelve or sixteen inches in diameter. The twenty sections of the circles can be made by use of a protractor: Each section being 18 degrees wide. It is suggested that the percents be lettered in blue and the fractions in red. When all the design work has been done, the surface should be shellacked so that it is more permanent and will soil less readily.



The Fracto-Percenter

The pointer should be made of wood with one end painted blue and the other end red. The blue end should be wide enough so that it will cover the fraction symbol in one section of the fraction circle. Note that the pointer is attached "off center" so that as it rotates the longer end (blue) covers a fraction and the red end points to the proper fraction. Attach the pointer with a short bolt at the center of the circles. Use a washer on the top surface so that the bolt head will not rub and so the pointer will turn easily but not swing too freely.

EDITOR'S NOTE. William Roys made the "Fracto-Percenter" last Spring when he was an eighth grader in Louise Mayer's class in Scarsdale. Teachers will see various uses of this device. It might be a worth-while class or homework project for pupils to make a similar device. They can learn a good deal as they plan, think, and carry out the work.

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